

# Curves of Constant Breadth According to Darboux Frame

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## Abstract

In this paper, we investigate constant breadth curves on a surface according to Darboux frame and give some characterizations of these curves.

*Key words and Phrases:* Darboux frame, constant breadth curve, Euclidean space.

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## 1 Introduction

Since the first introduction of constant breadth curves in the plane by L. Euler in 1778 [4], many researchers focused on this subject and found out a lot of properties about constant breadth curves in the plane [11], [3], [15]. Fujiwara has introduced constant breadth curves, by taking a closed curve whose normal plane at a point  $P$  has only one more point  $Q$  in common with the curve and for which the distance  $d(P, Q)$  is constant.

After the development of cam design, researchers have shown strong interest to this subject again and many interesting properties have been discovered. For example Köse has defined a new concept called space is a pair of curve of constant breadth in [9], a pair of unit speed space curves of class  $C^3$  with non-vanishing curvature and torsion in  $E^3$ , which have parallel tangents in opposite directions at corresponding points, and the distance between these points is always constant by using the Frenet frame.

The characterizations of Köse's paper on constant breadth curves in the space has led us to investigate this topic according to Darboux frame on a surface.

## 2 Basic Concepts

Now, we introduce some basic concepts about our study. Let  $M$  be an oriented surface and  $\beta$  be a unit speed curve of class  $C^3$  on  $M$ . As we know,  $\beta$  has a natural frame called Frenet frame  $\{T, N, B\}$  with properties below:

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned} \tag{1}$$

where  $\kappa$  is the curvature,  $\tau$  is the torsion,  $T$  is the unit tangent vector field,  $N$  is the principal normal vector field and  $B$  is the binormal vector field of the curve  $\beta$ .

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Let us take the unit tangent vector field of the curve  $\beta$  and the unit normal vector field  $n$  of the surface  $M$ . If we define unit vector field  $g$  as  $g = (n \circ \beta) \times T$  where  $\times$  cross product. We will have a new frame called Darboux frame  $\{T, g, n \circ \beta\}$ . The relations between these two frames can be given as follows:

$$\begin{bmatrix} T \\ g \\ n \circ \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (2)$$

where  $\alpha(s)$  is the angle between the vector fields  $n \circ \beta$  and  $B$ . If we take the derivatives of  $T, g, n$  with respect to  $s$ , we will have

$$\begin{bmatrix} T' \\ g' \\ (n \circ \beta)' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_g \\ -k_n & -t_g & 0 \end{bmatrix} \begin{bmatrix} T \\ g \\ n \circ \beta \end{bmatrix} \quad (3)$$

where  $k_g, k_n$  and  $t_g$  are called the geodesic curvature, the normal curvature and the geodesic torsion respectively. Then, we will have following relations

$$\begin{aligned} k_g &= \varkappa \cos \alpha \\ k_n &= \varkappa \sin \alpha \\ t_g &= \tau - \alpha' \end{aligned} \quad (4)$$

In the differential geometry of surfaces, for a curve  $\beta(s)$  lying on a surface, there are following cases:

- i)  $\beta$  is a geodesic curve if and only if  $k_g = 0$ .
- ii)  $\beta$  is an asymptotic line if and only if  $k_n = 0$ .
- iii)  $\beta$  is a principal line if and only if  $t_g = 0$ .

### 3 Curves of Constant Breadth According to Darboux Frame

Let  $\beta(s)$  and  $\beta^*(s^*)$  be a pair of unit speed curves of class  $C^3$  with non-vanishing curvature and torsion in  $E^3$  which have parallel tangents in opposite directions at corresponding points and the distance between these points is always constant. We will call  $(\beta^*, \beta)$  as curve pair of constant breath.

If  $\beta$  lies on a surface, it has Darboux frame in addition to Frenet frame with properties (1), (2), (3) and (4). So we may write for  $\beta^*$

$$\beta^*(s^*) = \beta(s) + m_1(s)T(s) + m_2(s)g(s) + m_3(s)(n \circ \beta)(s)$$

If we differentiate this equation with respect to  $s$  and use (3), we will have

$$\begin{aligned} (\beta^*)' &= \frac{d\beta^*}{ds^*} \frac{ds^*}{ds} = (1 + m_1' - m_3k_n - m_2k_g)T + (m_1k_g + m_2' - m_3t_g)g \\ &\quad + (m_1k_n + m_2t_g + m_3')(n \circ \beta) \end{aligned} \quad (5)$$

and

$$\frac{d\beta^*}{ds} = \frac{d\beta^*}{ds^*} \frac{ds^*}{ds} = T^* \frac{ds^*}{ds}$$

As we know  $\langle T, T^* \rangle = -1$ . Then

$$-\frac{ds^*}{ds} = 1 + m'_1 - m_3 k_n - m_2 k_g.$$

So we find from (5),

$$\begin{aligned} m'_1 &= m_2 k_g + m_3 k_n - 1 - \frac{ds^*}{ds} \\ m'_2 &= m_3 t_g - m_1 k_g \\ m'_3 &= -m_1 k_n - m_2 t_g. \end{aligned} \tag{6}$$

Let us denote the angle between the tangents at the points  $\beta(s)$  and  $\beta(s + \triangle s)$  with  $\triangle \theta$ . If we denote the vector  $T(s + \triangle s) - T(s)$  with  $\triangle T$ , we know  $\lim_{\triangle s \rightarrow 0} \frac{\triangle T}{\triangle s} = \lim_{\triangle s \rightarrow 0} \frac{\triangle \theta}{\triangle s} = \frac{d\theta}{ds} = \varkappa$ . We called the angle of contingency to the angle  $\triangle \theta$  [15]. Let us denote the differentiation with respect to  $\theta$  with  $\cdot$ . By using the equation  $\frac{d\theta}{ds} = \varkappa$ , we can write (6) as follows:

$$\begin{aligned} \dot{m}_1 &= \rho (m_2 k_g + m_3 k_n) - f(\theta) \\ \dot{m}_2 &= \rho (m_3 t_g - m_1 k_g) \\ \dot{m}_3 &= \rho (-m_1 k_n - m_2 t_g). \end{aligned} \tag{7}$$

where  $\rho = \frac{1}{\varkappa}$ ,  $\rho^* = \frac{1}{\varkappa^*}$  and  $\rho + \rho^* = f(\theta)$ .

Now we investigate curves of constant breadth according to Darboux frame for some special cases:

### 3.1 Case (For geodesic curves)

Let  $\beta$  be non straight line geodesic curve on a surface. Then  $k_g = \varkappa \cos \alpha = 0$  and  $\varkappa \neq 0$ , we get  $\cos \alpha = 0$ . So it implies that  $k_n = \varkappa$ ,  $t_g = \tau$ . By using (7), we have following differential equation system

$$\begin{aligned} \dot{m}_1 &= m_3 - f(\theta) \\ \dot{m}_2 &= m_3 \varphi \\ \dot{m}_3 &= -m_1 - m_2 \varphi \end{aligned} \tag{8}$$

where  $\varphi = \frac{\tau}{\varkappa}$ . By using (8), we obtain a differential equation as follows:

$$(\ddot{m}_1 + \ddot{f}) - \frac{d\varphi}{d\theta} \frac{1}{\varphi} (\dot{m}_1 + m_1 + \dot{f}) + (1 + \varphi^2) \dot{m}_1 + \varphi^2 f = 0 \tag{9}$$

We assume that  $(\beta^*, \beta)$  is a curve pair of constant breadth, then

$$\|\beta^* - \beta\|^2 = m_1^2 + m_2^2 + m_3^2 = \text{constant}$$

which implies that

$$m_1 \dot{m}_1 + m_2 \dot{m}_2 + m_3 \dot{m}_3 = 0 \quad (10)$$

By combining (8) and (10) then we get

$$m_1 f(\theta) = 0$$

### 3.1.1 Case $f(\theta) = 0$ .

We assume that  $f(\theta) = 0$ . By using (9), we get

$$\ddot{m}_1 - \frac{d\varphi}{d\theta} \frac{1}{\varphi} (\ddot{m}_1 + m_1) + (1 + \varphi^2) \dot{m}_1 = 0 \quad (11)$$

If  $\beta$  is a helix curve then  $\varphi = \varphi_0 = \text{constant}$ . From (11), we have

$$\ddot{m}_1 + (1 + \varphi_0^2) \dot{m}_1 = 0$$

whose solution is

$$m_1 = \frac{1}{\sqrt{1 + \varphi_0^2}} (c_1 \sin((1 + \varphi_0^2) \theta) - c_2 \cos((1 + \varphi_0^2) \theta))$$

So we can find as  $m_2 = -\frac{1}{\varphi_0} (m_1 + \ddot{m}_1)$  and  $m_3 = \dot{m}_1$ .

### 3.1.2 Case $m_1 = 0$ .

We assume that  $m_1 = 0$ , then by using (9), we get

$$\ddot{f} - \frac{d\varphi}{d\theta} \frac{1}{\varphi} \dot{f} + \varphi^2 f = 0 \quad (12)$$

If  $\beta$  is a helix curve, then  $\varphi = \varphi_0 = \text{constant}$ . From (12) we obtain

$$\ddot{f} + \varphi_0^2 f = 0$$

whose solution is

$$f(\theta) = c_1 \cos(\varphi_0 \theta) + c_2 \sin(\varphi_0 \theta)$$

Since  $m_1 = 0$  it implies that

$$m_3 = f(\theta)$$

$$m_2 = -\frac{\dot{m}_3}{\varphi_0}$$

**Theorem 1.** Let  $\beta$  be a geodesic curve and a helix curve. Let  $(\beta, \beta^*)$  be a pair of constant breadth curve. In that case  $\beta^*$  can be expressed as one of the following cases:  
i)

$$\beta^*(s^*) = \beta(s) + m_1(s)T(s) - \frac{1}{\varphi_0} (m_1(s) + \dot{m}_1(s))g(s) + \dot{m}_1(s)(n \circ \beta)(s)$$

where  $m_1 = \frac{1}{\sqrt{1+\varphi_0^2}} (c_1 \sin((1+\varphi_0^2)\theta) - c_2 \cos((1+\varphi_0^2)\theta))$ .

ii)

$$\beta^*(s^*) = \beta(s) - \frac{f(\theta)}{\varphi_0} g(s) + f(\theta) (n \circ \beta)(s)$$

where  $f(\theta) = c_1 \cos(\varphi_0 \theta) + c_2 \sin(\varphi_0 \theta)$ .

### 3.2 Case (For asymptotic lines)

Let  $\beta$  be non straight line asymptotic line on a surface. Then  $k_n = \varkappa \sin \alpha = 0$  and  $\varkappa \neq 0$ , we have  $\sin \alpha = 0$ . So we get  $k_g = \varepsilon \varkappa$ ,  $t_g = \tau$ , where  $\varepsilon = \pm 1$ . By using (7), we have following differential equation system

$$\begin{aligned} \dot{m}_1 &= \varepsilon m_2 - f(\theta) \\ \dot{m}_2 &= m_3 \varphi - \varepsilon m_1 \\ \dot{m}_3 &= -m_2 \varphi \end{aligned} \quad (13)$$

where  $\varphi = \frac{\tau}{\varkappa}$ . By using (13), we obtain a differential equation as follows:

$$(\ddot{m}_1 + \ddot{f}) - \frac{d\varphi}{d\theta} \frac{1}{\varphi} (\dot{m}_1 + m_1 + \dot{f}) + (1 + \varphi^2) \dot{m}_1 + \varphi^2 f = 0 \quad (14)$$

We assume that  $(\beta^*, \beta)$  is a curve pair of constant breadth then

$$\|\beta^* - \beta\|^2 = m_1^2 + m_2^2 + m_3^2 = \text{constant}$$

which implies that

$$m_1 \dot{m}_1 + m_2 \dot{m}_2 + m_3 \dot{m}_3 = 0 \quad (15)$$

By combining (13) and (15) then we get

$$m_1 f(\theta) = 0$$

#### 3.2.1 Case $f(\theta) = 0$

We assume that  $f(\theta) = 0$ . By using (14), we get

$$\ddot{m}_1 - \frac{d\varphi}{d\theta} \frac{1}{\varphi} (\dot{m}_1 + m_1) + (1 + \varphi^2) \dot{m}_1 = 0 \quad (16)$$

If  $\beta$  is a helix curve then  $\varphi = \varphi_0 = \text{constant}$ . From (16), we have

$$\ddot{m}_1 + (1 + \varphi_0^2) \dot{m}_1 = 0$$

whose solution is

$$m_1 = \frac{1}{\sqrt{1 + \varphi_0^2}} (c_1 \sin((1 + \varphi_0^2)\theta) - c_2 \cos((1 + \varphi_0^2)\theta))$$

So we can find as  $m_2 = \varepsilon \dot{m}_1$  and  $m_3 = \varepsilon \frac{1}{\varphi_0} (m_1 + \dot{m}_1)$ .

### 3.2.2 Case $m_1 = 0$

We assume that  $m_1 = 0$ . Then by using (14), we get

$$\ddot{f} - \frac{d\varphi}{d\theta} \frac{1}{\varphi} \dot{f} + \varphi^2 f = 0 \quad (17)$$

If  $\beta$  is a helix curve, then  $\varphi = \varphi_0 = \text{constant}$ . From (17) we obtain

$$\ddot{f} + \varphi_0^2 f = 0$$

whose solution is

$$f(\theta) = c_1 \cos(\varphi_0 \theta) + c_2 \sin(\varphi_0 \theta)$$

Since  $m_1 = 0$  it implies that

$$m_2 = \varepsilon f(\theta)$$

$$m_3 = \frac{\dot{m}_2}{\varphi_0}$$

where  $\theta = \int \kappa ds$ .

**Theorem 2.** Let  $\beta$  be an asymptotic line and a helix curve. Let  $(\beta, \beta^*)$  be a pair of constant breadth curve. In that case  $\beta^*$  can be expressed as one of the following cases:

i)

$$\beta^*(s^*) = \beta(s) + m_1(s)T(s) + \varepsilon \dot{m}_1(s)g(s) - \frac{\varepsilon}{\varphi_0} (m_1(s) + \dot{m}_1(s))(n \circ \beta)(s)$$

where  $m_1 = \frac{1}{\sqrt{1+\varphi_0^2}} (c_1 \sin((1+\varphi_0^2)\theta) - c_2 \cos((1+\varphi_0^2)\theta))$ .

ii)

$$\beta^*(s^*) = \beta(s) + \varepsilon f(\theta)g(s) + \varepsilon \frac{\dot{f}(\theta)}{\varphi_0} (n \circ \beta)(s)$$

where  $f(\theta) = c_1 \cos(\varphi_0 \theta) + c_2 \sin(\varphi_0 \theta)$ .

### 3.3 Case (For principal line)

We assume that  $\beta$  is a principal line. Then we have  $t_g = 0$  and it implies that  $\tau = \alpha'$ . By using (7), we get

$$\dot{m}_1 = m_2 \cos \alpha + m_3 \sin \alpha - f(\theta) \quad (18)$$

$$\dot{m}_2 = -m_1 \cos \alpha$$

$$\dot{m}_3 = -m_1 \sin \alpha$$

By using (18), we obtain following differential equation

$$(\ddot{m}_1 + \dot{m}_1) + (\dot{m}_1 + f)\dot{\alpha}^2 - \left( \sin \alpha \int m_1 \cos \alpha d\theta - \cos \alpha \int m_1 \sin \alpha d\theta \right) \ddot{\alpha} + \ddot{f} = 0 \quad (19)$$

since  $t_g = 0$  we obtain  $\dot{\alpha} = \frac{\tau}{\varepsilon}$ . We assume that  $(\beta^*, \beta)$  is a pair of curve is constant breadth. In that case

$$\|\beta^* - \beta\|^2 = m_1^2 + m_2^2 + m_3^2 = \text{constant}$$

which implies that

$$m_1 \dot{m}_1 + m_2 \dot{m}_2 + m_3 \dot{m}_3 = 0 \quad (20)$$

By combining (18) and (20) then we get

$$m_1 f(\theta) = 0$$

### 3.3.1 Case $f(\theta) = 0$

We assume that  $f(\theta) = 0$ . By using (19), we get

$$(\ddot{m}_1 + \dot{m}_1) + \dot{m}_1 \dot{\alpha}^2 - \left( \sin \alpha \int m_1 \cos \alpha d\theta - \cos \alpha \int m_1 \sin \alpha d\theta \right) \ddot{\alpha} = 0 \quad (21)$$

If  $\beta$  is a helix curve then  $\dot{\alpha} = \frac{\tau}{\varepsilon} = \text{constant}$ . From (21), we have

$$\ddot{m}_1 + (1 + \dot{\alpha}^2) \dot{m}_1 = 0$$

Then we get

$$m_1(s) = \frac{1}{\sqrt{1 + \dot{\alpha}^2}} (c_1 \sin((1 + \dot{\alpha}^2)\theta) - c_2 \cos((1 + \dot{\alpha}^2)\theta))$$

By using (18) we obtain

$$\begin{aligned} \dot{m}_2 &= -m_1 \cos \alpha \\ \dot{m}_3 &= -m_1 \sin \alpha \end{aligned}$$

where  $\alpha = \int \tau ds$ .

**Theorem 3.** Let  $\beta$  be a principal line and a helix curve. Let  $(\beta, \beta^*)$  be a pair of constant breadth curve such that  $\langle \beta^*, T \rangle = m_1 \neq 0$ . In that case  $\beta^*$  can be expressed as:

$$\beta^* = \beta + m_1(s)T(s) - m_1(s) \cos \alpha g(s) - m_1 \sin \alpha (n \circ \beta)(s)$$

where  $m_1(s) = \frac{1}{\sqrt{1 + \dot{\alpha}^2}} (c_1 \sin((1 + \dot{\alpha}^2)\theta) - c_2 \cos((1 + \dot{\alpha}^2)\theta))$ .

### 3.3.2 Case $m_1 = 0$

If  $m_1 = 0$ , then from (19)

$$\ddot{f} + \dot{\alpha}^2 f = 0 \quad (22)$$

where  $\dot{\alpha} = \frac{\tau}{\varepsilon}$ . On the other hand since  $m_1 = 0$  from (18) we have  $m_2 = c_2 = \text{constant}$ ,  $m_3 = c_3 = \text{constant}$  and

$$f = c_2 \cos \alpha + c_3 \sin \alpha \quad (23)$$

By combining (22) and (23)

$$\ddot{\alpha}(-c_2 \sin \alpha + c_3 \cos \alpha) = 0$$

In that case, if  $\ddot{\alpha} = 0$  then we obtain that  $\dot{\alpha} = \frac{\tau}{\kappa} = \text{constant}$ .  $\beta$  becomes a helix curve. If  $-c_2 \sin \alpha + c_3 \cos \alpha = 0$  then we have  $\alpha = \text{constant}$ . This means that  $\beta$  is a planar curve.

**Theorem 4.** *Let  $\beta$  be a principal line. Let  $(\beta, \beta^*)$  be a pair of constant breadth curve such that  $\langle \beta^*, T \rangle = m_1 = 0$ . In that case  $\beta$  is a helix curve or a planar curve and  $\beta^*$  can be expressed as:*

$$\beta^* = \beta + c_2 g(s) + c_3 (n \circ \beta)(s)$$

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